

Non-periodic geodesic ball packings to infinite regular prism tilings in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space *

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Abstract

In [14] we defined and described the *regular infinite or bounded* p -gonal prism tilings in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ space. We proved that there exist infinitely many regular infinite p -gonal face-to-face prism tilings $\mathcal{T}_p^i(q)$ and infinitely many regular bounded p -gonal non-face-to-face prism tilings $\mathcal{T}_p(q)$ for integer parameters p, q ; $3 \leq p, \frac{2p}{p-2} < q$. Moreover, in [5] and [7] we have determined the symmetry group of $\mathcal{T}_p(q)$ via its index 2 rotational subgroup, denoted by $\mathbf{pq2}_1$ and investigated the corresponding geodesic and translation ball packings.

In this paper we study the structure of the regular infinite or bounded p -gonal prism tilings, prove that the side curves of their base figures are arcs of Euclidean circles for each parameter. Moreover, we examine the non-periodic geodesic ball packings of congruent regular non-periodic prism tilings derived from the regular infinite p -gonal face-to-face prism tilings

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$\mathcal{T}_p^i(q)$ in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ geometry. We develop a procedure to determine the densities of the above non-periodic optimal geodesic ball packings and apply this algorithm to them. We look for those parameters p and q above, where the packing density large enough as possible. Now, we obtain larger density ≈ 0.626606 for $(p, q) = (29, 3)$ then the maximal density of the corresponding periodical geodesic ball packings under the groups $\mathbf{pq2}_1$.

In our work we will use the projective model of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ introduced by E. Molnár in [2].

1 Basic notions

The real 2×2 matrices $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$ with unit determinant $ad - bc = 1$ constitute a Lie transformation group by the usual product operation, taken to act on row matrices as on point coordinates on the right as follows

$$(z^0, z^1) \begin{pmatrix} d & b \\ c & a \end{pmatrix} = (z^0 d + z^1 c, z^0 b + z^1 a) = (w^0, w^1) \quad (1.1)$$

$$\text{with } w = \frac{w^1}{w^0} = \frac{b + \frac{z^1}{z^0} a}{d + \frac{z^1}{z^0} c} = \frac{b + za}{d + zc},$$

as action on the complex projective line \mathbf{C}^∞ (see [2], [3]). This group is a 3-dimensional manifold, because of its 3 independent real coordinates and with its usual neighbourhood topology ([9], [16], [8]). In order to model the above structure in the projective sphere \mathcal{PS}^3 and in the projective space \mathcal{P}^3 (see [2]), we introduce the new projective coordinates (x^0, x^1, x^2, x^3) where $a := x^0 + x^3$, $b := x^1 + x^2$, $c := -x^1 + x^2$, $d := x^0 - x^3$ with the positive, then the non-zero multiplicative equivalence as projective freedom in \mathcal{PS}^3 and in \mathcal{P}^3 , respectively. Then it follows that $0 > bc - ad = -x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3$ describes the interior of the above one-sheeted hyperboloid solid \mathcal{H} in the usual Euclidean coordinate simplex with the origin $E_0(1; 0; 0; 0)$ and the ideal points of the axes $E_1^\infty(0; 1; 0; 0)$, $E_2^\infty(0; 0; 1; 0)$, $E_3^\infty(0; 0; 0; 1)$. We consider the collineation group \mathbf{G}_* that acts on the projective sphere \mathcal{SP}^3 and preserves a polarity i.e. a scalar product of signature $(- - ++)$, this group leaves the one sheeted hyperboloid solid \mathcal{H} invariant. We have to choose an appropriate subgroup \mathbf{G} of \mathbf{G}_* as isometry group, then the universal covering group and space \mathcal{H} of \mathcal{H} will be the hyperboloid model of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ [2].

The specific isometries $\mathbf{S}(\phi)$ ($\phi \in \mathbf{R}$) constitute a one parameter group given by the matrices:

$$\mathbf{S}(\phi) : (s_i^j(\phi)) = \begin{pmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix} \quad (1.2)$$

The elements of $\mathbf{S}(\phi)$ are the so-called *fibre translations*. We obtain a unique fibre line to each $X(x^0; x^1; x^2; x^3) \in \tilde{\mathcal{H}}$ as the orbit by right action of $\mathbf{S}(\phi)$ on X . The coordinates of points lying on the fibre line through X can be expressed as the images of X by $\mathbf{S}(\phi)$:

$$(x^0; x^1; x^2; x^3) \xrightarrow{\mathbf{S}(\phi)} (x^0 \cos \phi - x^1 \sin \phi; x^0 \sin \phi + x^1 \cos \phi; x^2 \cos \phi + x^3 \sin \phi; -x^2 \sin \phi + x^3 \cos \phi). \quad (1.3)$$

The points of a fibre line through X by usual inhomogeneous Euclidean coordinates $x = \frac{x^1}{x^0}$, $y = \frac{x^2}{x^0}$, $z = \frac{x^3}{x^0}$, $x^0 \neq 0$ are given by

$$(1; x; y; z) \xrightarrow{\mathbf{S}(\phi)} \left(1; \frac{x + \tan \phi}{1 - x \tan \phi}; \frac{y + z \tan \phi}{1 - x \tan \phi}; \frac{z - y \tan \phi}{1 - x \tan \phi}\right) \quad (1.4)$$

for the projective space \mathcal{P}^3 , where ideal points (at infinity) conventionally occur.

In (1.3) and (1.4) we can see the 2π periodicity of ϕ , moreover the (logical) extension to $\phi \in \mathbf{R}$, as real parameter, to have the universal covers $\tilde{\mathcal{H}}$ and $\widetilde{\mathbf{SL}_2\mathbf{R}}$, respectively, through the projective sphere \mathcal{PS}^3 . The elements of the isometry group of $\mathbf{SL}_2\mathbf{R}$ (and so by the above extension the isometries of $\widetilde{\mathbf{SL}_2\mathbf{R}}$) can be described by the matrix (a_i^j) (see [2] and [3]) Moreover, we have the projective proportionality, of course. We define the *translation group* \mathbf{G}_T , as a subgroup of the isometry group of $\mathbf{SL}_2\mathbf{R}$, the isometries acting transitively on the points of \mathcal{H} and by the above extension on the points of $\widetilde{\mathbf{SL}_2\mathbf{R}}$ and $\tilde{\mathcal{H}}$. \mathbf{G}_T maps the origin $E_0(1; 0; 0; 0)$ onto $X(x^0; x^1; x^2; x^3)$. These isometries and their inverses (up to a positive determinant factor) can be given by the following matrices:

$$\mathbf{T} : (t_i^j) = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ -x^1 & x^0 & x^3 & -x^2 \\ x^2 & x^3 & x^0 & x^1 \\ x^3 & -x^2 & -x^1 & x^0 \end{pmatrix}. \quad (1.5)$$

The rotation about the fibre line through the origin $E_0(1; 0; 0; 0)$ by angle ω ($-\pi < \omega \leq \pi$) can be expressed by the following matrix (see [2])

$$\mathbf{R}_{E_0}(\omega) : (r_i^j(E_0, \omega)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \omega & \sin \omega \\ 0 & 0 & -\sin \omega & \cos \omega \end{pmatrix}, \quad (1.6)$$

and the rotation $\mathbf{R}_X(\omega)$ about the fibre line through $X(x^0; x^1; x^2; x^3)$ by angle ω can be derived by formulas (1.5) and (1.6):

$$\mathbf{R}_X(\omega) = \mathbf{T}^{-1} \mathbf{R}_{E_0}(\omega) \mathbf{T} : (r_i^j(X, \omega)). \quad (1.7)$$

Horizontal intersection of the hyperboloid solid \mathcal{H} with the plane $E_0 E_2^\infty E_3^\infty$ provides the *hyperbolic \mathbf{H}^2 base plane* of the model $\widetilde{\mathcal{H}} = \widetilde{\mathbf{SL}_2 \mathbf{R}}$. The fibre through X intersects the base plane $z^1 = x = 0$ in the foot point

$$Z(z^0 = x^0 x^0 + x^1 x^1; z^1 = 0; z^2 = x^0 x^2 - x^1 x^3; z^3 = x^0 x^3 + x^1 x^2). \quad (1.8)$$

We introduce a so-called hyperboloid parametrization by [2] as follows

$$\begin{aligned} x^0 &= \cosh r \cos \phi, & x^1 &= \cosh r \sin \phi, \\ x^2 &= \sinh r \cos(\theta - \phi), & x^3 &= \sinh r \sin(\theta - \phi), \end{aligned} \quad (1.9)$$

where (r, θ) are the polar coordinates of the base plane and ϕ is just the fibre coordinate. We note that

$$-x^0 x^0 - x^1 x^1 + x^2 x^2 + x^3 x^3 = -\cosh^2 r + \sinh^2 r = -1 < 0.$$

The inhomogeneous coordinates corresponding to (1.9), that play an important role in the later visualization of prism tilings in \mathbf{E}^3 , are given by

$$\begin{aligned} x &= \frac{x^1}{x^0} = \tan \phi, & y &= \frac{x^2}{x^0} = \tanh r \frac{\cos(\theta - \phi)}{\cos \phi}, \\ z &= \frac{x^3}{x^0} = \tanh r \frac{\sin(\theta - \phi)}{\cos \phi}. \end{aligned} \quad (1.10)$$

1.1 Geodesic balls in $\widetilde{\mathbf{SL}_2 \mathbf{R}}$

Definition 1.1 *The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2 .*

Definition 1.2 The geodesic sphere of radius ρ (denoted by $S_{P_1}(\rho)$) with the center in point P_1 is defined as the set of all points P_2 with the condition $d(P_1, P_2) = \rho$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection.

Definition 1.3 The body of the geodesic sphere of centre P_1 and with radius ρ is called geodesic ball, denoted by $B_{P_1}(\rho)$, i.e., $Q \in B_{P_1}(\rho)$ iff $0 \leq d(P_1, Q) \leq \rho$.

From [5] it follows that $S(\rho)$ is a simply connected surface in \mathbf{E}^3 and $\widetilde{\mathbf{SL}_2\mathbf{R}}$, respectively, if $\rho \in [0, \frac{\pi}{2})$. If $\rho \geq \frac{\pi}{2}$ then the universal cover should be discussed. Therefore, we consider geodesic spheres and balls only with radii $\rho \in [0, \frac{\pi}{2})$ in the following.

1.2 The volume of a geodesic ball

The volume formula of the geodesic ball $B(\rho)$ follows from the metric tensor g_{ij} (see [5]). We obtain the connection between the hyperboloid coordinates (r, θ, ϕ) and the geographical coordinates (s, λ, α) in a standard way. Therefore, the volume of the geodesic ball of radius ρ can be computed by the following

Theorem 1.1

$$\begin{aligned} \text{Vol}(B(\rho)) &= \int_B \frac{1}{2} \sinh(2r) \, dr \, d\theta \, d\phi = \\ &= 4\pi \int_0^\rho \int_0^{\frac{\pi}{4}} \frac{1}{2} \sinh(2r(s, \alpha)) |J_1| \, d\alpha \, ds \\ &\quad + 4\pi \int_0^\rho \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2} \sinh(2r(s, \alpha)) |J_2| \, d\alpha \, ds \end{aligned} \quad (1.11)$$

where $|J_1| = \left| \begin{array}{cc} \frac{\partial r}{\partial s} & \frac{\partial r}{\partial \alpha} \\ \frac{\partial \phi}{\partial s} & \frac{\partial \phi}{\partial \alpha} \end{array} \right|$ and similarly $|J_2|$ (by Table 1 and $\frac{\partial \theta}{\partial \lambda} = 1$) are the corresponding Jacobians.

The complicated formulas above need numerical approximations by computer.

1.3 Regular bounded periodic prism tilings and their space groups $\text{pq}2_1$

In [14] we have defined and described the regular prisms and prism tilings with a space group class $\Gamma = \text{pq}2_1$ of $\widetilde{\mathbf{SL}_2\mathbf{R}}$. These will be summarized in this section.

Definition 1.4 Let \mathcal{P}^i be an infinite solid that is bounded by certain surfaces that can be determined (in [14]) by „side fibre lines” passing through the vertices of a regular p -gon \mathcal{P}^b lying in the base plane. The images of solids \mathcal{P}^i by $\widetilde{\mathbf{SL}_2\mathbf{R}}$ isometries are called infinite regular p -sided prisms. Here regular means that the side surfaces are congruent to each other under rotations about a fiber line (e.g. through the origin).

The common part of \mathcal{P}^i with the base plane is the *base figure* of \mathcal{P}^i that is denoted by \mathcal{P} and its vertices coincide with the vertices of \mathcal{P}^b , **but \mathcal{P} is not assumed to be a polygon.**

Definition 1.5 A bounded regular p -sided prism is analogously defined if the face of the base figure \mathcal{P} and its translated copy \mathcal{P}^t , under a fibre translation by (1.2) and so (1.3), are also introduced. The faces \mathcal{P} and \mathcal{P}^t are called cover faces.

We consider regular prism tilings $\mathcal{T}_p(q)$ by prisms $\mathcal{P}_p(q)$ where q pieces regularly meet at each side edge by q -rotation.

The following theorem has been proved in [14]:

Theorem 1.2 There exist regular bounded not face-to-face prism tilings $\mathcal{T}_p(q)$ in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ for each $3 \leq p \in \mathbb{N}$ where $\frac{2p}{p-2} < q \in \mathbb{N}$.

We assume that the prism $\mathcal{P}_p(q)$ is a *topological polyhedron* having at each vertex one p -gonal cover face (it is not a polygon at all) and two *skew quadrangles* which lie on certain side surfaces in the model. Let $\mathcal{P}_p(q)$ be one of the tiles of $\mathcal{T}_p(q)$, \mathcal{P}^b is centered in the origin with vertices $A_1 A_2 A_3 \dots A_p$ in the base plane (Fig. 1 and 2). It is clear that the side curves $c_{A_i A_{i+1}}$ ($i = 1 \dots p$, $A_{p+1} \equiv A_1$) of the base figure are derived from each other by $\frac{2\pi}{p}$ rotation about the vertical x axis, so there are congruent in $\widetilde{\mathbf{SL}_2\mathbf{R}}$ sense. The corresponding vertices $B_1 B_2 B_3 \dots B_p$ are generated by a fibre translation τ given by (1.3) with parameter $0 < \Phi \in \mathbb{R}$. The fibre lines through the vertices $A_i B_i$ are denoted by f_i , ($i = 1, \dots, p$) and the fibre line through the ”midpoint” H of the curve $c_{A_1 A_p}$ is denoted by f_0 . This f_0 will be a half-screw axis as follows below.

The tiling $\mathcal{T}_p(q)$ is generated by a discrete isometry group $\Gamma_p(q) = \mathbf{pq2}_1 \subset \text{Isom}(\widetilde{\mathbf{SL}_2\mathbf{R}})$ which is given by its fundamental domain $A_1 A_2 O A_1^s A_2^s O^s$ a *topological polyhedron* and the group presentation (see Fig. 1 and 4 for $p = 3$ and [14] for details):

$$\begin{aligned} \mathbf{pq2}_1 &= \{a, b, s : a^p = b^q = asa^{-1}s^{-1} = babs^{-1} = 1\} = \\ &= \{a, b : a^p = b^q = ababa^{-1}b^{-1}a^{-1}b^{-1} = 1\}. \end{aligned} \quad (1.12)$$

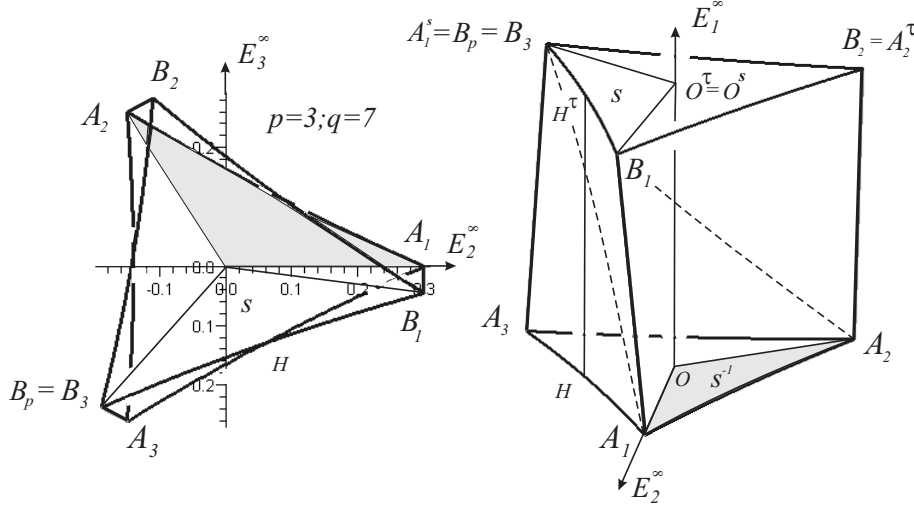


Figure 1: The regular prism $\mathcal{P}_p(q)$ and the fundamental domain of the space group $\mathbf{pq2}_1$

Here \mathbf{a} is a p -rotation about the fibre line through the origin (x axis), \mathbf{b} is a q -rotation about the fibre line trough A_1 and $\mathbf{s} = \mathbf{bab}$ is a screw motion $\mathbf{s} : OA_1A_2 \rightarrow O^sB_pB_1$. All these can be obtained by formulas (1.5) and (1.6). Then we get that $\mathbf{abab} = \mathbf{baba} =: \tau$ is a fibre translation. Then \mathbf{ab} is a 2_1 half-screw motion about $f_0 = HH^\tau$ (look at Fig. 1) that also determines the fibre translation τ above. This group in (3.1) surprisingly occurred in § 6 of our paper [6] at double links $K_{p,q}$. The coordinates of the vertices $A_1A_2A_3 \dots A_p$ of the base figure and the corresponding vertices $B_1B_2B_3 \dots B_p$ of the cover face can be computed for all given parameters p, q by

$$\tanh(OA_1) = b := \sqrt{\frac{1 - \tan \frac{\pi}{p} \tan \frac{\pi}{q}}{1 + \tan \frac{\pi}{q} \tan \frac{\pi}{q}}}. \quad (1.13)$$

1.4 The volume of the bounded regular prisms

The volume formula of a *sector-like* 3-dimensional domain $Vol(D(\Psi))$ can standardly be computed by the metric tensor g_{ij} (see [5]). in hyperboloid coordinates. This defined by the base figure D lying in the base plane and by fibre translation τ given by (1.3) with the height parameter Ψ .

Theorem 1.3 Suppose we are given a sector-like region D , so a continuous function $r = r(\theta)$ where the radius r depends upon the polar angle θ . The volume of domain $D(\Psi)$ is derived by the following integral:

$$\begin{aligned} \text{Vol}(D(\Psi)) &= \int_D \frac{1}{2} \sinh(2r(\theta)) dr d\theta d\psi = \\ &= \int_0^\Psi \int_{\theta_1}^{\theta_2} \int_0^{r(\theta)} \frac{1}{2} \sinh(2r(\theta)) dr d\theta d\psi = \Psi \int_{\theta_1}^{\theta_2} \frac{1}{4} (\cosh(2r(\theta)) - 1) d\theta. \end{aligned} \quad (1.14)$$

$\mathcal{P}_p(q)$ be an arbitrary bounded regular prism. We get the following

Theorem 1.4 The volume of the bounded regular prism $\mathcal{P}_p(q)$ ($3 \leq p \in \mathbb{N}$, $\frac{2p}{p-2} < q \in \mathbb{N}$) can be computed by the following simple formula:

$$\text{Vol}(\mathcal{P}_p(q)) = \text{Vol}(D(p, q, \Psi)) \cdot p, \quad (1.15)$$

where $\text{Vol}(D(p, q, \Psi))$ is the volume of the sector-like 3-dimensional domain that is given by the sector region $OA_1A_2 \subset \mathcal{P}$ (see Fig. 1 and 3) and by Ψ the $\widetilde{\text{SL}_2\mathbb{R}}$ height of the prism, depending on p, q .

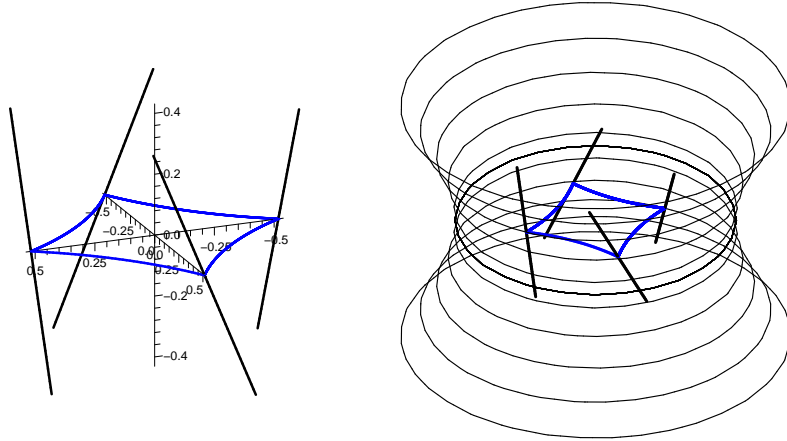


Figure 2: Regular infinite 4-gonal prism $\mathcal{P}_4^i(6)$ of the infinite regular prism tiling $\mathcal{T}_4^i(6)$

2 Regular infinite prism tilings and non-periodic ball packings

2.1 Infinite regular prism tilings

In this subsection we study the regular infinite prism tilings $\mathcal{T}_p^i(q)$. Let $\mathcal{T}_p(q)$ be a regular prism tiling and let $\mathcal{P}_p(q)$ be one of its tiles which is given by its base figure \mathcal{P} that is centered at the origin K with vertices $G_1G_2G_3 \dots G_p$ in the base plane of the model and the corresponding vertices $A_1A_2A_3 \dots A_p$ and $B_1B_2B_3 \dots B_p$ are generated by fibre translations $-\tau$ and τ given by (1.3) with parameter $\Psi = \frac{\pi}{2} - \frac{\pi}{p} - \frac{\pi}{q}$. The images of the topological polyhedron $\mathcal{P}_p(q)$ by the translations $\langle \tau \rangle$ form an infinite prism $\mathcal{P}_p^i(q)$ (see Definitions 1. 4-5). By the

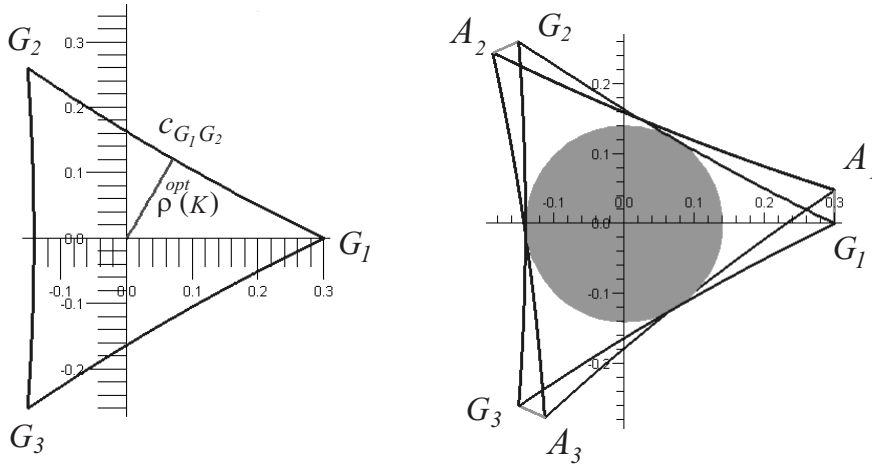


Figure 3: The maximal radius $\rho^{opt}(K)$ and the optimal half prism $A_1A_2A_3G_1G_2G_3$ with optimal half sphere for parameters $p = 3$, $q = 7$ with the maximal radius

constuction of the bounded prism tilings follows that rotations through $\omega = \frac{2\pi}{q}$ about the fibre lines f_i maps the corresponding side face onto the neighbouring one. Therefore, we have got the following (see [14]):

Theorem 2.1 *There exist regular infinite face-to-face prism tilings $\mathcal{T}_p^i(q)$ for integer parameters p, q where $3 \leq p$, $\frac{2p}{p-2} < q$.*

For example, we have described $\mathcal{P}_4^i(6)$ with its base polygon in Fig. 2, where the parameter $b = \frac{\sqrt{6}-\sqrt{2}}{2}$.

2.2 Non-periodic geodesic ball packings

We consider a infinite regular prism tiling $\mathcal{T}_p^i(q)$ and let $\mathcal{P}_p^i(q)$ one of its tiles with base figure \mathcal{P} centered at the origin with vertices $G_1G_2 \dots G_p$ in the base plane of the model. Let B_K^{opt} be the geodesic ball with center at the origin K that touches the side surfaces of the infinite regular prism $\mathcal{P}_p^i(q)$. The radius of the ball B_K^{opt} is denoted by $\rho^{opt}(K)$. Moreover, we define the regular prism $\mathcal{P}_p^{opt}(q) = A_1A_2 \dots A_pB_1B_2 \dots B_p$ with base figure \mathcal{P} and with cover faces $A_1A_2 \dots A_p$ and $B_1B_2 \dots B_p$ touching B_K^{opt} . It is clear, that the height $h_p^{opt}(q)$ of $\mathcal{P}_p^{opt}(q)$ is $2\rho^{opt}(K)$.

The images of $\mathcal{P}_p^{opt}(q)$ by the fibre traslations $\langle \tau \rangle$ where $h_p^{opt}(q) = |\tau| = 2\rho^{opt}(K)$ cover the infinite regular prism $\mathcal{P}_p^i(q)$ and by the structure of the infinite prism tilings follows that rotations through $\omega = \frac{2\pi}{q}$ about the fibre lines f_i maps the corresponding side face onto the neighbouring one and thus the images of $\mathcal{P}_p^{opt}(q)$ fill the $\widetilde{\text{SL}_2\mathbf{R}}$ space without overlap. These tilings are denoted by $\mathcal{T}_p^n(q)$.

The height $h_p^{opt}(q)$ of the prism $\mathcal{P}_p^{opt}(q)$ is not equal to $\pi - \frac{2\pi}{p} - \frac{2\pi}{q}$ so the corresponding regular prism tiling is non-periodic. We note here, that there are infinitely many non-periodic prism tilings derived from $\mathcal{T}_p^n(q)$.

For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $\mathcal{P}_p^{opt}(q)$. The density of the optimal ball packing of the prism tiling $\mathcal{T}_p^n(q)$ ($3 \leq p$, $\frac{2p}{p-2} < q$, integer parameters) can be computed by the following formula:

$$\delta_p^{opt}(q) := \frac{\text{Vol}(B_K^{opt})}{\text{Vol}(\mathcal{P}_p^{opt}(q))}.$$

In order to determine the optimal radius $\rho^{opt}(K)$ we will use the following Lemmas. The equation of the side curve $c_{G_1G_2}$ is derived as the foot points (see (1.3) and (1.8)) of the corresponding fibre lines ($3 \leq p$, $\frac{2p}{p-2} < q$, where p and q are integer parameters):

Lemma 2.2 *The parametric equation of the side curve $c_{G_1G_2}$ of the base figure \mathcal{P}*

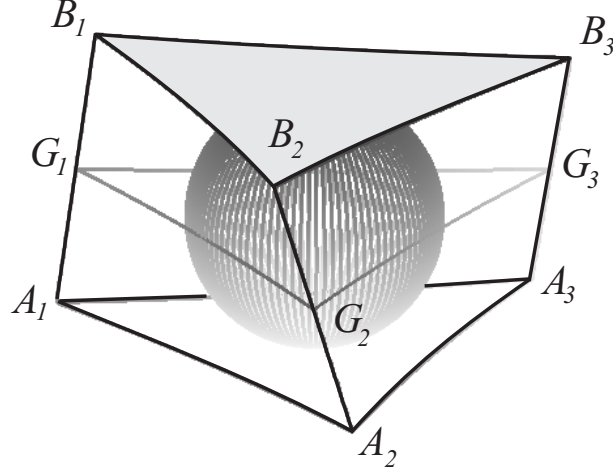


Figure 4: The optimal prism $A_1A_2A_3B_1B_2B_3$ with optimal sphere for parameters $p = 3, q = 7$ with the maximal radius $\rho^{opt}(K)$

is

$$\begin{aligned}
 c_p^q(t) = & \left(0, \sqrt{\sin\left(\frac{2\pi}{p} + \frac{2\pi}{q}\right)} \left(t \cos\left(\frac{2\pi}{p}\right) \sin^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) - \frac{t}{2} \sin\left(\frac{2\pi}{p}\right) \sin\left(\frac{2\pi}{p} + \frac{2\pi}{q}\right) + \right. \right. \\
 & \left. \left. \sin^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) (1-t) + t^2 \cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \cos\left(\frac{\pi}{p} - \frac{\pi}{q}\right) \right) / \right. \\
 & \left. \left(\sqrt{\left(\sin\left(\frac{2\pi}{p}\right) + \sin\left(\frac{2\pi}{q}\right)\right)} \left(\sin^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) + t^2 \cos^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \right) \right), \right. \\
 & t \sqrt{\sin\left(\frac{2\pi}{p} + \frac{2\pi}{q}\right)} \left(\sin\left(\frac{2\pi}{p}\right) \sin^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) + \frac{1}{2} \cos\left(\frac{2\pi}{p}\right) \sin\left(\frac{2\pi}{p} + \frac{2\pi}{q}\right) (1-t) + \right. \\
 & \left. \left. \cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \left(t \sin\left(\frac{2\pi}{p}\right) \cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right) + \sin\left(\frac{\pi}{p} + \frac{\pi}{q}\right) (t-1) \right) \right) / \right. \\
 & \left. \left(\sqrt{\left(\sin\left(\frac{2\pi}{p}\right) + \sin\left(\frac{2\pi}{q}\right)\right)} \left(\sin^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) + t^2 \cos^2\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \right) \right), \right. \\
 & \left. t \in [0, 1] \right). \tag{2.1}
 \end{aligned}$$

The side curves $c_{G_i G_{i+1}}$ ($i = 1 \dots p$, $G_{p+1} \equiv G_1$) of the base figure are derived from each other by $\frac{2\pi}{p}$ rotation about the vertical x axis, so there are congruent and their curvatures are equal in $\widetilde{\text{SL}_2\mathbf{R}}$ sense. Moreover, the above side curves are congruent also in Euclidean sense, therefore their curvatures are equal in Euclidean sense, as well. We obtain by the usual machinery of the differential geometry the next

Lemma 2.3 *The curvature $C_p(q)$ of the side curves $c_{G_i G_{i+1}}$ ($i = 1 \dots p$, $G_{p+1} \equiv G_1$) in the Euclidean sense is*

$$C_p(q) = \sqrt{\frac{\cos\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \left(\sin\left(\frac{2\pi}{p}\right) + \sin\left(\frac{2\pi}{q}\right)\right)}{\sin\left(\frac{\pi}{p} + \frac{\pi}{q}\right) \left(1 - \cos\left(\frac{2\pi}{p}\right)\right)}} \quad (2.2)$$

therefore, the side curves $c_{G_i G_{i+1}}$ ($i = 1 \dots p$, $G_{p+1} \equiv G_1$) are Euclidean circular arcs of radius $r_p^q = \frac{1}{C_p(q)}$.

Remark 2.1 1. *It is easy to see, that the asymptotic behaviour of $C_p(q)$ is the following: $\lim_{q \rightarrow \infty}(C_p(q)) = \cot\left(\frac{\pi}{p}\right)$, $\lim_{p \rightarrow \infty}(C_p(q)) = \infty$.*

2. *Given a point off of a line, if we drop a perpendicular to the above line from the given point, then x is the distance along this perpendicular segment, and let $\phi = \Pi(x)$ is the least angle such that the line drawn through the point at that angle does not intersect the given line. The angle ϕ is the angle of parallelism. By the famous formel of J. Bolyai follows, that $\log(\cot(\phi)) = x$. Therefore, if we denote the distance of parallelism of the angle ϕ by $\Lambda(\phi)$ then $\log\left(\lim_{q \rightarrow \infty}(C_p(q))\right) = \log\left(\cot\left(\frac{\pi}{p}\right)\right) = \Lambda\left(\frac{\pi}{p}\right)$.*

In the Table 1 we have collected some values of the radii of curvature r_3^q of the side curve $c_{G_1 G_2}$ of the base figur \mathcal{P} .

Table 1				
(p, q)	(3, 7)	(3, 8)	(3, 10)	(3, 1000)
$C_p(q)$	0.286926	0.371579	0.453885	0.577339
r_p^q	3.485219	2.691215	2.203203	1.732085

The maximal radius $\rho^{opt}(K)$ of the balls B_K^{opt} can be determined using the above Lemmas for all possible parameters as the distance between the origin and $c_{G_1 G_2}$. The volumes $Vol(B_K^{opt})$ can be computed by the Theorem 1.3 and the volumes of the prisms $\mathcal{P}_p^{opt}(q)$ can be determined by the Theorem 1.4.

The above locally densest geodesic ball packings can be determined for all regular prism tilings $\mathcal{T}_p^n(q)$ (p, q as above). We have summarized in the following Tables some results to tilings $\mathcal{T}_p^n(q)$.

Table 2				
(p, q)	$\rho^{opt}(K)$	$Vol(B_K^{opt})$	$Vol(\mathcal{P}_p^{opt}(q))$	$\delta_p^{opt}(q)$
(3, 7)	0.141564	0.011963	0.031767	0.376592
(3, 8)	0.181760	0.025431	0.071377	0.356287
(3, 10)	0.219795	0.045198	0.138101	0.327281
(3, 1000)	0.274648	0.088981	0.428828	0.207499
\vdots	\vdots	\vdots	\vdots	\vdots
(4, 5)	0.265319	0.080085	0.166705	0.480397
(4, 6)	0.329239	0.154965	0.344779	0.449464
(4, 10)	0.404230	0.292043	0.761956	0.383280
(4, 1000)	0.440683	0.382228	1.378910	0.277196
\vdots	\vdots	\vdots	\vdots	\vdots
(5, 4)	0.313435	0.133256	0.246171	0.541312
(5, 5)	0.421241	0.332010	0.661684	0.501765
(5, 10)	0.530638	0.686600	1.667047	0.411866
(5, 1000)	0.562086	0.825191	2.639937	0.312580
\vdots	\vdots	\vdots	\vdots	\vdots
(6, 4)	0.440687	0.382237	0.692229	0.552183
(6, 5)	0.530638	0.686600	1.333638	0.514833
(6, 10)	0.629251	1.188024	2.767592	0.429263
(6, 1000)	0.658476	1.377893	4.124915	0.334042
\vdots	\vdots	\vdots	\vdots	\vdots
(7, 3)	0.272637	0.087010	0.142753	0.609513
(7, 4)	0.535202	0.705586	1.261041	0.559527
(7, 5)	0.617496	1.117400	2.133913	0.523639
(7, 10)	0.710652	1.772033	4.018646	0.440953
(7, 1000)	0.738668	2.015812	5.785244	0.348440
\vdots	\vdots	\vdots	\vdots	\vdots
(8, 3)	0.382143	0.245334	0.400179	0.613062
(8, 4)	0.612113	1.086117	1.923010	0.564800
(8, 5)	0.690221	1.608804	3.035751	0.529953
(8, 10)	0.780165	2.422804	5.392115	0.449324
(8, 1000)	0.807443	2.722797	7.589676	0.358750
\vdots	\vdots	\vdots	\vdots	\vdots

Table 3				
(p, q)	$\rho^{opt}(K)$	$Vol(B_K^{opt})$	$Vol(\mathcal{P}_p^{opt}(q))$	$\delta_p^{opt}(q)$
(10, 3)	0.530638	0.686600	1.111365	0.617799
\vdots	\vdots	\vdots	\vdots	\vdots
(20, 3)	0.914848	4.195479	6.706186	0.625613
(20, 4)	1.094612	8.023914	13.755306	0.583332
(20, 5)	1.163424	10.092704	18.275027	0.552268
(20, 10)	1.245625	13.132701	27.392724	0.479423
(20, 1000)	1.271043	14.216772	35.858024	0.396474
\vdots	\vdots	\vdots	\vdots	\vdots
(28, 3)	1.088398	7.855861	12.537440	0.626592
(29, 3)	1.106311	8.348310	13.323054	0.626606
(30, 3)	1.123593	8.847342	14.119487	0.626605
\vdots	\vdots	\vdots	\vdots	\vdots
(35, 3)	1.201914	11.432334	18.250297	0.626419
\vdots	\vdots	\vdots	\vdots	\vdots
(40, 3)	1.269482	14.148085	22.599777	0.626028
\vdots	\vdots	\vdots	\vdots	\vdots
(52, 3)	1.401728	21.089811	33.761388	0.624673
\vdots	\vdots	\vdots	\vdots	\vdots
(72, 3)	1.565173	33.642710	54.088487	0.621994

Remark 2.2 1. The best density that we found ≈ 0.626606 for parameters $p = 29, q = 3$ that is larger then the maximal density of the corresponding periodical geodesic ball packings under the groups $\mathbf{pq2}_1$.

2. The problems of finding the densest geodesic and translation ball packings in the Thurston geometries are timely (see e.g. [4], [10], [11], [12], [13]).

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